

THE THEORY OF DISCONJUGACY FOR A SECOND ORDER LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. This is an introduction to the theory of disconjugacy for a second order linear differential equation. We give new proofs of some of basic results and obtain new sufficient conditions for disconjugacy (in particular, on the whole real axis).

The differential equation

$$(1) \quad (Lx)(t) := x'' + p(t)x' + q(t)x = 0,$$

is called *disconjugate* on an open interval $J \subset \mathbb{R}$ if any of its non-trivial solutions have at most one zero in J . The property of disconjugacy, which guarantees the existence of the unique solution to the boundary value problem

$$x'' + p(t)x' + q(t)x = f(t) \quad , \quad x(a) = 0, \quad x(b) = 0,$$

was discovered in 1951 by A. Wintner [4], and since then attracted the interest of many mathematicians [5]–[13], in particular due to its great importance to qualitative theory of differential equation (1).

In this article we describe the present state of the theory of disconjugacy for differential equation (1) (Sections 1–8).

We also obtain a new sufficient condition for disconjugacy of (1) (Sections 9 and 10). Traditionally (see, e.g., [1] — [13]), the conditions for disconjugacy are obtained for differential equations of the form

$$(2) \quad x'' + Q(t)x = 0$$

and include the assumptions of smallness of coefficient Q . Our condition does not necessarily require the smallness of q .

1. We start with recalling some definitions. Let us consider differential equation

$$(3) \quad (Lx)(t) = f(t)$$

where

$$t \in I := (a, b), \quad -\infty \leq a < b \leq +\infty, \quad p, q, f : I \mapsto \mathbb{R} \text{ are locally summable.}$$

A function $x : I \rightarrow \mathbb{R}$ is called *solution* of equation (3) if it has locally absolutely continuous first derivative x' and satisfies equation (3) almost everywhere (with respect to Lebesgue measure). Under our assumptions on p, q there exists the unique solution of equation (3) satisfying $x(a) = \xi_0, x'(a) = \xi_1, a \in I$. Recall that the general solution of (1) has form $x(t) = c_1 u_1(t) + c_2 u_2(t)$,

where u_1, u_2 are linearly independent solutions of (1), c_1, c_2 are arbitrary constants; the pair $\{u_1, u_2\}$ is called *fundamental system* of (1); its Wronskian $W(t) := [u_1, u_2](t)$ is nowhere zero on I .

Function $C : I \times [\alpha, t] \rightarrow \mathbb{R}$ is called *Cauchy's function* of equation (1) if

$$(LC)(\cdot, s) = 0 \quad \text{for almost all } t \geq s, \quad C(s, s) = 0, \quad \frac{\partial C(s, s)}{\partial t} = 1 \quad (s \in I).$$

We note that Cauchy's function always exists and is unique. One can represent the general solution of equation (3) in the form

$$(4) \quad x(t) = c_1 u_1(t) + c_2 u_2(t) + \int_a^t C(t, s) f(s) ds$$

where (u_1, u_2) is the fundamental system of solutions of (1) and c_1, c_2 are arbitrary constants.

Function $G : [a, b]^2 \rightarrow \mathbb{R}$ is called *Green's function* of boundary value problem

$$(5) \quad (Lx)(t) = f(t) \quad (t \in I), \quad x(a) = 0, \quad x(b) = 0 \quad (a, b \in I),$$

provided that it satisfies the following conditions:

- 1) G is continuous on $[a, b]^2$;
- 2) $\frac{\partial G(\cdot, s)}{\partial t}$ is absolutely continuous in the triangles $a \leq s < t \leq b$ and $a \leq t < s \leq b$, and

$$\frac{\partial G(s+, s)}{\partial t} - \frac{\partial G(s-, s)}{\partial t} = 1;$$

$$3) (LG)(\cdot, s) = 0 \quad t \neq s;$$

$$4) G(a, s) = 0, \quad G(b, s) = 0.$$

If boundary value problem (5) has the unique solution, then it has the unique Green's function, and its solution x admits presentation

$$x(t) = \int_a^b G(t, s) f(s) ds.$$

Also, one has the following identity

$$G(t, s) = \begin{cases} -\frac{C(b, t)C(s, a)}{C(b, a)}, & \text{if } a \leq s < t, \\ -\frac{C(t, a)C(b, s)}{C(b, a)}, & \text{if } t \leq s \leq b, \end{cases}$$

which implies that if $C(t, s) > 0$, for $a \leq s < t \leq b$, then $G(t, s) < 0$ for $(t, s) \in (a, b)^2$.

2. We will also need the following two results due to Sturm: Separation of Zeros Theorem and Comparison Theorem [14, p. 252], [15, p. 81].

Theorem 1 (Separation of Zeros). Let $a, b \in I$, suppose that x is a solution of equation (1) such that $x(a) = x(b) = 0$, $x(t) \neq 0$ for any $t \in (a, b)$. Then any other solution of (1), linearly independent with x , has the only zero in (a, b) .

Proof. Suppose that y is a solution of equation (1) linearly independent with x and such that $y(t) \neq 0$ on (a, b) . Since $y(a) \neq 0$, $y(b) \neq 0$ (due to linear independence of x and y), $y(t) \neq 0$ on $[a, b]$. Therefore, function

$$h(t) := -\frac{W(t)}{y(t)},$$

where Wronskian W of $\{x, y\}$ is continuous and nowhere zero on $[a, b]$, hence $h(t) \neq 0$ on $[a, b]$. Without loss of generality $h(t) > 0$ on $[a, b]$. Since

$$h = \left(\frac{x}{y} \right)',$$

we have $\int_a^b h(t) dt > 0$ and, at the same time,

$$\int_a^b h(t) dt = \frac{x(b)}{y(b)} - \frac{x(a)}{y(a)} = 0.$$

The latter implies that $y(t^*) = 0$ at some $t^* \in (a, b)$.

If $y(t_*) = 0$ at some $t_* \neq t^*$, then, as we already proved, x would have a zero in (a, b) , which contradicts to our assumptions. \square

Let $a \in I$, x be a solution of equation (1) such that $x(a) = 0$. Point $\rho_+(a) > a$ ($\rho_-(a) < a$) is called *right (left) conjugate point of a* if

$$x(\rho_{\pm}(a)) = 0, \quad x(t) \neq 0 \quad \text{in} \quad (a, \rho_+(a)) \quad (\text{in } (\rho_-(a), a)).$$

If $x(t) \neq 0$ on (a, β) (respectively, (α, a)), we define $\rho_+(a) = \beta$ ($\rho_-(a) = \alpha$)

Corollary 1. *Functions ρ_{\pm} are strictly increasing. Furthermore, $\rho_+(\rho_-(t)) = \rho_-(\rho_+(t)) = t$ ($t \in I$), i.e., functions ρ_{\pm} are the inverses of each other and map continuously any interval in I to an interval in I .*

Proof. Let $t_2 > t_1$, $x(t_1) = y(t_2) = 0$ (x and y are solutions of (1)). Suppose that $\rho_+(t_2) \leq \rho_+(t_1)$. The equality here, meaning that $x(\rho_+(t_2)) = y(\rho_+(t_1)) = 0$, contradicts to the definition of a conjugate point. Meanwhile, the strict inequality contradicts to Theorem 1 (since y would have two zeros between two consecutive zeros of x .) Consequently, $\rho_+(t_2) > \rho_+(t_1)$.

The proof for ρ_- is similar. The proof of the second statement follows from the definition of conjugate points and properties of strictly monotone functions. \square

Definition 1. Differential equation (1) is called *disconjugate* on an interval $J \subset I$ if any of its non-trivial solutions has at most one zero in J . (We also say that J is an interval of disconjugacy of equation (1)).

Thus, J is an interval of disconjugacy of equation (1) if and only if $\rho_{\pm}(a) \notin J$ for any $a \in J$.

In what follows, we write $L \in \mathfrak{I}(J)$ if equation (1) is disconjugate on interval $J \subset I$.

Let $(a, b) \subset I$, suppose that $a_n \rightarrow a+$, $b_n \rightarrow b-$ ($a_n \rightarrow -\infty$, $b_n \rightarrow +\infty$ in the case $a = \alpha = -\infty$, $b = \beta = +\infty$). Then

$$(6) \quad \mathfrak{I}((a, b)) = \bigcap_{n=1}^{\infty} \mathfrak{I}([a_n, b_n]) = \bigcap_{n=1}^{\infty} \mathfrak{I}((a_n, b_n)).$$

As follows from the definitions of the property of disconjugacy and Cauchy's function, if equation (1) is disconjugate on the interval $J = [a, b] \subset I$, then $C(t, s) > 0$ in the triangle $a \leq s < t < b$. Disconjugacy of equation (1) on an interval $[a, b]$ implies the existence of the unique solution of problem (5), so Green's function of this problem satisfies $G(t, s) < 0$ on $(a, b)^2$.

Theorem 2 (Comparison Theorem). Let

$$L_i y := y'' + p(t) y' + q_i(t) y = 0, \quad i = 1, 2 \quad q_1(t) \leq q_2(t) \quad (t \in I).$$

If $L_2 \in \mathfrak{T}(J)$, then $L_1 \in \mathfrak{T}(J)$.

Proof. Let $L_2 \in \mathfrak{T}(J)$. Suppose that $L_1 \notin \mathfrak{T}(J)$. Then there exist a solution x of equation $L_1 x = 0$, and points $a, b \in J$, $a < b$, $x(a) = x(b) = 0$, $x(t) > 0$ ($t \in (a, b)$). Let us define the solution y of equation $L_2 y = 0$ by initial values $y(a) = 0$, $y'(a) = x'(a) (> 0)$. Since $L_2 \in \mathfrak{T}(J)$, then $y(b) > 0$. Let $h := y - x$. Then $0 = L_2 y - L_1 x = L_2 h - (q_1 - q_2)x$, $h(a) = 0$, $h'(a) = 0$. Also, note that $h(b) > 0$. Thus, function h is a solution to problem

$$(L_2 h)(t) = (q_1(t) - q_2(t))x(t), \quad h(a) = h'(a) = 0,$$

i.e.,

$$h(t) = \int_a^t C_2(t, s)(q_1(s) - q_2(s))x(s) ds \leq 0,$$

since Cauchy's functions $C_2(t, s)$ of equation $L_2 y = 0$ is positive for all $a \leq s < t \leq b$, $x(s) > 0$, $(q_1(s) - q_2(s)) \leq 0$ for all $a < s \leq t \leq b$. The inequality $h(b) \leq 0$ contradicts to the inequality $h(b) > 0$ obtained previously, so $L_1 \in \mathfrak{T}(J)$. \square

3. We use Theorem 1 and Corollary 1 to prove the following statement.

Theorem 3. Let $a, b \in I$. Then

$$\mathfrak{T}((a, b)) = \mathfrak{T}([a, b]) = \mathfrak{T}([a, b)).$$

Proof. It suffices to check inclusion $\mathfrak{T}((a, b)) \subset \mathfrak{T}([a, b])$, the rest follows by symmetry.

Let $L \in \mathfrak{T}((a, b))$. Suppose that $L \notin \mathfrak{T}([a, b])$. Then there exist a solution v of equation (1) such that $v(a) = v(c) = 0$ ($a < c < b$), $v(t) > 0$ in (a, c) (note that v has at least two zeros in $[a, b]$, and at most one zero in (a, b)). By definition, $c = \rho_+(a)$ ($(a = \rho_-(c))$). Let us choose $c_1 \in (c, b)$ so that $v(t) < 0$ in (c, c_1) . We put $a_1 = \rho_-(c_1)$. According to Corollary 1 $a < a_1 < c$. Let x be the corresponding solution of equation (1), i.e., $x(c_1) = x(a_1) = 0$, $x(t) > 0$ in (a_1, c_1) . Since $L \in \mathfrak{T}([a_1, c_1])$, there exists the unique solution y of equation (1) that satisfies $y(a_1) = y(c_1) = 1$. We have $y(t) > 0$ on $[a_1, c_1]$ (as a continuous function taking the same values at the endpoints of the interval, function y can have only even number of zeros, hence, due to disconjugacy, none of them). We note that y is linearly independent with v and with x . According to Theorem 1 y has exactly one zero in both intervals (a, a_1) and (c, c_1) , that is, y has two zeros in (a, b) . The latter contradicts to disconjugacy of equation (1) on (a, b) . \square

Theorem 4. 1. If there exists a solution of equation (1) that is nowhere zero on $[a, b] \subset I$ ($((a, b) \subset I)$), then $L \in \mathfrak{T}([a, b])$ ($L \in \mathfrak{T}((a, b))$).

2. If $L \in \mathfrak{T}([a, b]$, $[a, b] \subset I$) ($L \in \mathfrak{T}([a, b])$, $[a, b] \subset I$), then there exists a solution of equation (1) that is nowhere zero on $[a, b]$ ($((a, b) \subset I)$).

Proof. 1. The statement follows immediately from Theorem 1.

2. Let $J = [a, b]$, $[a, b] \subset I$. Let us determine solutions $y_1(t)$ and $y_2(t)$ by initial conditions $y_1(a) = 0$, $y_1'(a) = 1$ and $y_2(b) = 0$, $y_2'(b) = -1$. Since $L \in \mathfrak{T}([a, b])$, one has

$$y_1(t) > 0 \quad (t \in (a, b]), \quad y_2(t) > 0 \quad (t \in [a, b)).$$

The solution $y_1(t) + y_2(t)$ is the one required.

If $L \in \mathfrak{T}([a, b))$, then the required solution is y_1 . \square

It is possible that there are no solutions preserving sign on $[a, b)$. For instance,

$$L := \frac{d^2}{dt^2} + 1 \in \mathfrak{T}([0, \pi)).$$

However, any solution of equation $Lx = 0$ has precisely one zero in $[0, \pi)$.

4. Below we prove two theorems which demonstrate the role of disconjugacy in the theory of differential equation (1). This is Factorization Theorem (i.e., the theorem on representation of L as the product of linear differential operators of the first order [1],[3]) and generalized Rolle's Theorem ([16, p. 63]).

Theorem 5 (Factorization Theorem). Suppose $J = [a, b] \subset I$ or $J = (a, b) \subset I$. One has $L \in \mathfrak{T}(J)$, if and only if there exist functions $h_i, i = 0, 1, 2$ such that h'_0, h_1 are absolutely continuous, h_2 is summable on J , $h_i(t) > 0$, $h_0(t)h_1(t)h_2(t) \equiv 1$ on J , and

$$(7) \quad (Lx)(t) = h_2(t) \frac{d}{dt} h_1(t) \frac{d}{dt} h_0(t) x(t) \quad (t \in J, \quad x' \text{ absolutely continuous on } J).$$

Proof. Necessity. Let $L \in \mathfrak{T}(J)$. According to Theorem 4 there exists a solution y of equation (1) such that $y(t) > 0$ on J . Let u be a solution of equation (1) linearly independent with y and such that $w(t) := [y, u](t) > 0$. Let us consider the following linear differential operator of the second order

$$\widehat{L}x := \frac{w}{y} \frac{dt}{dt} \frac{y^2}{w} \frac{dt}{dt} \frac{x}{y}.$$

Since functions y, u form a fundamental system of solutions of both equation (1) and equation $\widehat{L}x = 0$, the top coefficient in \widehat{L} is equal to $\frac{w}{y} \frac{y^2}{w} \frac{1}{y} \equiv 1$, then $Lx \equiv \widehat{L}x$. These conditions are satisfied if $h_0 = \frac{1}{y}$, $h_1 = \frac{y^2}{w}$, $h_2 = \frac{w}{y}$.

Sufficiency. Suppose that we have identity (7). Then function $y(t) := \frac{1}{h_0(t)} > 0$ ($t \in J$) is a solution of equation (1) satisfying conditions of Theorem 4 which, in turn, implies that $L \in \mathfrak{T}(J)$. \square

Theorem 6 (Generalized Rolle's Theorem). Let $J = [a, b] \subset I$ or $J = (a, b) \subset I$, $L \in \mathfrak{T}(J)$. Suppose that function u has absolutely continuous on J first derivative, and function Lu is continuous. If there exist m ($m \geq 2$) geometrically distinct zeros of u in J , then Lu has at least $m - 2$ geometrically distinct zeros in J .

Proof. According to Theorem 5 one has representation (7). Since function $h_0 u$ has m geometrically distinct zeros in J , by Rolle's Theorem both $\frac{d}{dt} h_0 u$ and $h_1 \frac{d}{dt} h_0 u$ have at least $m - 1$ geometrically distinct zeros in J . Also according to Rolle's Theorem Lu has at least $m - 2$ geometrically distinct zeros in J . \square

5. In this section we provide some criteria for disconjugacy based on Theorems 2 and 4.

Criterion 1. Let $I = (-\infty, +\infty)$, $p(t) \equiv p = \text{const}$, $q(t) \equiv q = \text{const}$. Then differential equation (1) having constant coefficients $p(t) \equiv p$, $q(t) \equiv q$ is disconjugate on I if and only if the roots of its characteristic equation $\lambda^2 + p\lambda + q = 0$ are real.

Proof. Let ν be a real root of the characteristic equation. Then function $x(t) := e^{\nu t}$ is a solution of equation (1) nowhere vanishing on I . According to the first statement of Theorem 4, equation (1) is disconjugate on I .

Conversely, let (1) be disconjugate on I . Suppose that the characteristic equation has roots $\gamma \pm \delta i, \delta \neq 0$. Then solution $x(t) = e^{\gamma t} \cos \delta t$ of equation (1) has infinitely many zeros in I , which contradicts to its disconjugacy on I . \square

Let us consider equation

$$(8) \quad x'' + \frac{p}{t}x' + q(t)x = 0 \quad (t \in I(0, +\infty), \quad \text{where } p = \text{const.})$$

Criterion 2. If $q(t) \leq \frac{(p-1)^2}{4t^2}$, then equation (8) is disconjugate on $I := (0, +\infty)$.

Proof. Euler equation $x'' + \frac{p}{t}x' + \frac{(p-1)^2}{4t^2}x = 0$ is disconjugate on I by Theorem 4 since it has solution $x(t) = t^{\frac{1-p}{2}}$, which is nowhere equal to zero on I (let us also take into account (6)). According to Theorem 2 equation (8) is also disconjugate on this interval. \square

The next sufficient condition of disconjugacy is due to A.M. Lyapunov [12].

Criterion 3. Let $p(t) \equiv 0$, $q(t) \geq 0$ and $\int_a^b q(t)dt \leq \frac{4}{b-a}$. Then $L \in \mathfrak{T}([a, b])$.

Proof. Suppose that equation (1) possesses a non-trivial solution $y(t)$ having two zeros in $[a, b]$. Since y can not have multiple roots, we may assume, without loss of generality, that

$$(9) \quad y(a) = y(b) = 0.$$

Function y , as a solution of boundary value problem (1), (9), satisfies the following integral equation

$$(10) \quad y(t) = - \int_a^b G(t, s)q(s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} -\frac{(b-t)(s-a)}{b-a}, & \text{if } a \leq s < t, \\ -\frac{(t-a)(b-s)}{b-a}, & \text{if } t \leq s \leq b \end{cases}$$

is Green's function of equation $y'' = 0$ with boundary conditions (9). It is immediate that for $t \neq s$

$$(11) \quad |G(t, s)| < \frac{(b-s)(s-a)}{b-a}.$$

Let $\max_{s \in [a, b]} |y(s)| = |y(t^*)|$. Then (10) and (11)

$$\begin{aligned} |y(t^*)| &= \left| \int_a^b G(t^*, s)q(s)y(s)ds \right| \leq \int_a^b |G(t^*, s)| |y(s)|q(s)ds < \\ &< |y(t^*)| \int_a^b \frac{(b-s)(s-a)q(s)}{b-a}ds \leq \frac{b-a}{4} \int_a^b q(s)ds \end{aligned}$$

since $(b-s)(s-a) \leq \frac{(b-a)^2}{4}$ for $s \in [a, b]$. Therefore, $1 < \frac{b-a}{4} \int_a^b q(s)ds$, which contradicts to the conditions of the theorem. \square

Corollary 2. If $p(t) \equiv 0$,

$$\int_a^b q_+(t) dt \leq \frac{4}{b-a},$$

then $L \in \mathfrak{T}([a, b])$ ($q_+(t) = q(t)$ if $q(t) > 0$, $q_+(t) = 0$ if $q(t) \leq 0$).

Proof. As we already proved, $L_+ := \frac{d^2}{dt^2} + q_+ \in \mathfrak{T}([a, b])$. At the same time, since $q(t) \leq q_+(t)$, one has $L \in \mathfrak{T}([a, b])$. \square

Remark 1. We note that constant 4 in formulation of Criterion 3 is sharp.

The latter follows from the next example. Suppose that function v is twice continuously differentiable on $[0, 1]$ and

$$v(t) = t \quad (0 \leq t \leq \frac{1}{2} - \delta), \quad v(t) = 1 - t \quad \text{if } t > \frac{1}{2} + \delta,$$

$$v(t) > 0, \quad v''(t) < 0 \quad \text{if } \frac{1}{2} - \delta < t < \frac{1}{2} + \delta.$$

Define

$$q(t) = \begin{cases} -\frac{v''(t)}{v(t)}, & \text{if } t \in (0, 1), \\ 0, & \text{if } t = 0, t = 1. \end{cases}$$

Clearly, q is continuous, $q(t) \geq 0$ on $[0, 1]$; $L := \frac{d^2}{dt^2} + q(t) \notin \mathfrak{T}([0, 1])$, since equation $Ly = 0$ has solution $y = v(t)$ which has two zeros in $[0, 1]$. However,

$$\frac{v''}{v} = \left(\frac{v'}{v}\right)' + \left(\frac{v'}{v}\right)^2 \geq \left(\frac{v'}{v}\right)',$$

so the value of integral

$$\int_0^1 q(t) dt = - \int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} \left(\frac{v'}{v}\right)' dt = -\frac{v'}{v} \Big|_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} = \frac{4}{1-2\delta}$$

can be made arbitrarily close to 4 by choosing sufficiently small δ .

6. Criterion 4 is an example of a *non-effective* criterion of disconjugacy, i.e., a criterion formulated rather in terms of solutions of equation (1) than in terms of the coefficients of this equation.

Let us now give a necessary and sufficient condition of disconjugacy of equation (1). This criterion may be called *semi-effective* [10] (it is effective as a necessary condition, but non-effective as a sufficient condition). Although it is not expressed in terms of the coefficients of equation (1), it can be used to obtain sufficient conditions of disconjugacy formulated in terms of the coefficients of the equation. The author of this criterion is Valle-Poussin [2].

Theorem 7. Let $[a, b] \subset I$. One has $L \in \mathfrak{T}([a, b])$ if and only if there exists function v having first derivative absolutely continuous on $[a, b]$ and such that

$$(12) \quad v(t) > 0 \quad (a < t \leq b), \quad Lv \leq 0 \quad \text{a.e. on } [a, b].$$

Proof. Necessity follows from Theorem 4. Let us show that the conditions of the theorem are sufficient. In the case $v(a) = 0$ let us put $\tilde{v}(t) = v(t) + \varepsilon u(t)$, where $\varepsilon > 0$, and $u(t)$ is the solution of equation (1) with initial conditions $u(a) = 1$, $u'(a) = 0$. For ε sufficiently small we have $\tilde{v}(t) > 0$ on $[a, b]$. Hence we may assume, without loss of generality, that $v(t) > 0$ on $[a, b]$. Let us consider equation

$$(13) \quad Mx := x'' + px' - \frac{v'' + pv'}{v}x = 0.$$

According to Theorem 4 $M \in \mathfrak{T}([a, b])$ (since equation (13) has solution v positive on $[a, b]$). By our assumptions $v''(t) + p(t)v'(t) + q(t)v(t) \leq 0$, i.e., $-\frac{v''(t) + p(t)v'(t)}{v(t)} \geq q(t)$, a.e. on $[a, b]$. The statement of the theorem now follows from Theorem 2. \square

The proof of the next statement follows the same argument.

Theorem 8. If there exists function v having first derivative absolutely continuous on $[a, b]$ and such that

$$(14) \quad v(t) > 0 \quad (a < t < b), \quad Lv \leq 0 \quad \text{a.e. on } (a, b),$$

then $L \in \mathfrak{T}([a, b])$.

7. By choosing a particular ‘test’ function v we can get various effective conditions for disconjugacy.

Criterion 4. If $q(t) \leq 0$ on $[a, b] \subset I$ ($(a, b) \subset I$), then $L \in \mathfrak{T}([a, b])$ ($L \in \mathfrak{T}((a, b))$).

Proof. We put $v(t) \equiv 1$ and then use Theorem 7 (Theorem 8). \square

Criterion 5. Suppose that $p(t) = O(t - a)$ if $t \rightarrow a+$, $p(t) = O(b - t)$ if $t \rightarrow b-$ (in particular, $p(t) \equiv 0$). If

$$\frac{\pi}{b-a} \cot \frac{\pi(t-a)}{b-a} p(t) + q(t) \leq \frac{\pi^2}{(b-a)^2},$$

then $L \in \mathfrak{T}([a, b])$.

Proof. Let us choose $v(t) \equiv \sin \frac{\pi(t-a)}{b-a}$ and then use Theorems 8 ad 3. \square

Criterion 6. Suppose that we have inequality

$$(15) \quad |p(t)| \cdot \left| \frac{b+a}{2} - t \right| + |q(t)| \cdot \frac{(b-t)(t-a)}{2} \leq 1$$

or inequality

$$(16) \quad \frac{b-a}{2} \operatorname{essup}_{t \in (a,b)} |p(t)| + \frac{(b-a)^2}{8} \operatorname{essup}_{t \in (a,b)} |q(t)| \leq 1.$$

Then $L \in \mathfrak{T}([a, b])$.

Proof. Indeed, we take $v(t) \equiv \frac{(b-t)(t-a)}{2}$ and then refer to Theorems 3 and 8. \square

Let us note that inequality (16) implies inequality (15). Let $P(t, \lambda) := \lambda^2 + p(t)\lambda + q(t)$ be the ‘characteristic’ polynomial.

Criterion 7. If there exists $\nu \in \mathbb{R}$ such that $P(t, \nu) \leq 0$ ($t \in (-\infty, +\infty)$), then equation (1) is disconjugate on $(-\infty, +\infty)$.

Proof. One has $v(t) := e^{\nu t} > 0$ and $(Lv)(t) = e^{\nu t} P(t, \nu) \leq 0$ on $(-\infty, +\infty)$. The rest follows from Theorem 7. \square

8. Let us now formulate criteria that can be obtained from Theorem 7 (Theorem 8) using a ‘test’ function depending on coefficients of equation (1).

1°. Let us consider equation

$$(17) \quad \tilde{L}x := x'' + Px' + Qx = 0$$

having constant coefficients P and Q , in assumption that it is disconjugate on $[a, b]$. Let v be the solution of boundary value problem $\tilde{L}v = -1$, $v(a) = v(b) = 0$, let $\tilde{C}(t, s)$ be Cauchy’s function of equation (17). Then

$$\tilde{C}(t, s) > 0 \quad (a \leq s < t < b) \quad \text{and} \quad v(t) = \int_a^b M(t, s) ds > 0 \quad (t \in (a, b)),$$

where

$$M(t, s) := \begin{cases} \frac{\tilde{C}(b, t) \cdot \tilde{C}(s, a)}{\tilde{C}(b, a)}, & a \leq s \leq t \leq b, \\ \frac{\tilde{C}(t, a) \cdot \tilde{C}(b, s)}{\tilde{C}(b, a)}, & a \leq t < s \leq b \end{cases} > 0, \quad (t, s) \in ((a, b) \times (a, b)).$$

Since $(Lv)(t) = -1 + (p(t) - P)v'(t) + (q(t) - Q)v(t)$, inequality $(Lv)(t) \leq 0$ is satisfied if

$$(18) \quad (p(t) - P) \int_a^b \frac{\partial M(t, s)}{\partial t} ds + (q(t) - Q) \int_a^b M(t, s) ds \leq 1, \quad t \in (a, b).$$

As a result, we get the following statement.

Criterion 8. If (18) holds, then (1) is disconjugate on $[a, b]$.

The special choice of coefficients P and Q can lead to criteria for disconjugacy that are more subtle than the ones formulated above.

2°. Consider the particular case $Q = 0$. We have

$$M(t, s) = \begin{cases} \frac{(1 - e^{-P(b-t)})(1 - e^{-P(s-a)})}{P(1 - e^{-P(b-a)})} & (s \leq t), \\ \frac{(1 - e^{-P(t-a)})(1 - e^{-P(b-s)})}{P(1 - e^{-P(b-a)})} & (s > t). \end{cases}$$

It is immediate that

$$\begin{aligned} v(t) &= \frac{(1 - e^{-P(b-t)})(t - a - \frac{1}{P}(1 - e^{-P(t-a)}))}{P(1 - e^{-P(b-a)})} + \\ &+ \frac{(1 - e^{-P(t-a)})(b - t - \frac{1}{P}(1 - e^{-P(b-t)}))}{P(1 - e^{-P(b-a)})} \leq \frac{2(\frac{b-a}{2} - \frac{1}{P}(1 - e^{-P\frac{b-a}{2}}))}{P(1 + e^{-P\frac{b-a}{2}})}, \\ v'(t) &= \frac{P((b-t)e^{-P(t-a)} - (t-a)e^{-P(b-t)}) + e^{-P(b-t)} - e^{-P(t-a)}}{P(1 - e^{-P(b-a)})}, \\ |v'(t)| &\leq \frac{|P(b-a) + e^{-P(b-a)} - 1|}{P(1 - e^{-P(b-a)})}. \end{aligned}$$

Since condition $Lv \leq 0$ is now equivalent to inequality $(p(t) - P)v'(t) + q(t)v(t) \leq 1$, we get the following criterion.

Criterion 9. If

$$|p(t) - P| \frac{|P(b-a) + e^{-P(b-a)} - 1|}{P(1 - e^{-P(b-a)})} + |q(t)| \frac{2(\frac{b-a}{2} - \frac{1}{P}(1 - e^{-P\frac{b-a}{2}}))}{P(1 + e^{-P\frac{b-a}{2}})} \leq 1$$

($a < t < b$), then equation (1) is disconjugate on $[a, b]$.

3°. If we take, instead of auxiliary equation (17), equation $\tilde{L}x := x'' + p(t)x' = 0$, and take as v the solution of problem $\tilde{L}v = -1$, $v(a) = v(b) = 0$, we obtain the following criterion.

Criterion 10. If $q(t) \int_a^b M(t, s) ds \leq 1$, $t \in (a, b)$, where

$$M(t, s) = \begin{cases} \frac{\int_t^b e^{-\int_t^\sigma p(\mu) d\mu} d\sigma \cdot \int_a^s e^{-\int_a^\sigma p(\mu) d\mu} d\sigma}{\int_a^b e^{-\int_a^\sigma p(\mu) d\mu} d\sigma} & (s \leq t), \\ \frac{\int_a^t e^{-\int_a^\sigma p(\mu) d\mu} d\sigma \cdot \int_s^b e^{-\int_s^\sigma p(\mu) d\mu} d\sigma}{\int_a^b e^{-\int_a^\sigma p(\mu) d\mu} d\sigma} & (t < s), \end{cases}$$

then equation (1) is disconjugate on $[a, b]$.

9. Let us now consider a second order criterion for disconjugacy on the whole real axis \mathbb{R} . Let us consider differential equation

$$(19) \quad \tilde{L}x := x'' + px' + qx = 0$$

having constant coefficients p and q . As was shown before (see Criterion 1), disconjugacy of equation (19) on \mathbb{R} is equivalent to inequality $p^2 - 4q \geq 0$.

We will associate to equation (19) the point $\tilde{\mathcal{L}} = (p, q)$ in p, q -plane Π . Let

$$\mathfrak{N} := \{(p, q) : p^2 - 4q \geq 0\}, \quad \mathfrak{D} := \mathbb{R}^2 \setminus \mathfrak{N}.$$

Then according to Criterion 1

$$\tilde{L} \in \mathfrak{T}((-\infty, +\infty)) \iff \tilde{\mathcal{L}} \in \mathfrak{N}.$$

Let us now consider differential equation

$$(20) \quad Lx := x'' + p(t)x' + q(t)x = 0$$

with coefficients continuous on $(-\infty, +\infty)$. Every equation of form (20) gives rise to Jordan curve $G_L = \{t : (p(t), q(t))\}$ in plane Π . More precisely, it determines the motion DG_L along this curve.

Consider the following problem: *under which conditions the inclusion $G_L \subset \mathfrak{N}$ guaranteed disconjugacy of equation (20) on \mathbb{R} ?*

Below we formulate several possible (and simple) answers to this question.

A. Let $p(t) \equiv p = \text{const}$. Then inclusion $G_L \subset \mathfrak{N}$ is equivalent to inequality $q(t) \leq \frac{1}{4}p^2$. Function $v(t) := e^{-\frac{p}{2}t} > 0$ ($t \in \mathbb{R}$) satisfies inequality

$$(Lv)(t) = e^{-\frac{p}{2}t} \left(q(t) - \frac{1}{4}p^2 \right) \leq 0 \quad (t \in \mathbb{R}).$$

Hence, if $p(t) \equiv p = \text{const}$, $q(t) \leq \frac{1}{4}p^2$, then equation (20) is disconjugate on \mathbb{R} .

B. Let G_L be a line or a segment, suppose $G_L \subset \mathfrak{N}$. The equation of such line has form either $q(t) \equiv q = \text{const} \leq 0$ (for any $p(t)$) or $p = p(t)$, $q = -\gamma^2 + kp(t)$, where $|k| \leq \gamma$ ($\gamma > 0$) (in the case $k = \pm\gamma$ the line touches parabola $q = \frac{1}{4}p^2$). In the first case disconjugacy of equation (20)

on \mathbb{R} follows from Theorem 2. In the second case function $v(t) := e^{-kt} > 0$ ($t \in \mathbb{R}$) satisfies inequality $(Lv)(t) = e^{-kt}(k^2 - \gamma^2) \leq 0$ ($t \in \mathbb{R}$). Therefore, if G_L is a line in Π contained in \mathfrak{N} , or a segment of such a line, then equation (20) is disconjugate on \mathbb{R} .

C. Let $\gamma \geq 0$. Let us define

$$\mathfrak{M}_{\pm}(\gamma) = \{(p, q) : q \leq -\gamma^2 \pm \gamma p\}$$

Since both lines $q = -\gamma^2 \pm \gamma p$ touch parabola $q = \frac{1}{4}p^2$, then $\mathfrak{M}_{\pm}(\gamma) \subset \mathfrak{N}$ for any $\gamma \geq 0$.

Using the statements of sections **A** and **B** and Theorem 2, we get the following theorem.

Theorem 9. If for a certain $\gamma \geq 0$

$$G_L \subset \mathfrak{M}_+(\gamma) \quad (G_L \subset \mathfrak{M}_-(\gamma)),$$

then equation (20) is disconjugate on \mathbb{R} .

$$(\text{Put } v(t) = e^{-\gamma t} \quad (v(t) = e^{\gamma t}).)$$

We note that conditions of Theorem 9 depend only on curve G_L but not on motion DG_L along this curve.

D. The conditions of the statements below now *depend* on motion DG_L .

Theorem 10. Suppose that $r : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, p is a differentiable function, and one of the following conditions is satisfied:

$$(21) \quad p'(t) \geq 2r(t) \quad (p'(t) \leq -2r(t)) \quad (t \in \mathbb{R})$$

or

$$(22) \quad p^2(t) - 4p'(t) + r(t) \leq 0 \quad (p^2(t) + 4p'(t) + r(t) \leq 0) \quad (t \in \mathbb{R}).$$

Also, let $q(t) \leq \frac{p^2(t)}{4} + r(t)$ ($t \in \mathbb{R}$). Then equation (20) is disconjugate on \mathbb{R} .

Proof. Suppose that we have the first inequality (21). Under our assumptions, consider differential equation

$$(23) \quad L_2x := x'' + p(t)x' + \left(\frac{p^2(t)}{4} + r(t)\right)x$$

Put

$$v(t) = e^{-\frac{1}{2} \int_0^t p(s) ds}; \quad \text{then } v(t) > 0, \quad (L_2v)(t) = \left(-\frac{1}{2}p'(t) + r(t)\right) e^{-\frac{1}{2} \int_0^t p(s) ds} \leq 0, \quad t \in \mathbb{R}.$$

Consequently, $L_2 \in \mathfrak{T}((-\infty, +\infty))$. Disconjugacy of equation (20) now follows from Theorem 2.

If we have the second inequality (21), then, making substitution $y(t) = x(-t)$, we arrive to equation

$$y'' - p(t)y' + q(t)y = 0$$

and expression

$$L_2y := y'' - p(t)y' + \left(\frac{p^2(t)}{4} + r(t)\right)y,$$

for which we define $v(t) = e^{\frac{1}{2} \int_0^t p(s) ds}$. The rest repeats the argument above.

Suppose that the first inequality (22) is satisfied. Now we put

$$v(t) = e^{-\int_0^t p(s) ds} (> 0).$$

It is easy to see that

$$(L_2 v)(t) = \left(p^2(t) - \frac{1}{2} p'(t) - p^2(t) + \frac{p^2(t)}{4} + r(t) \right) e^{-\frac{1}{2} \int_0^t p(s) ds} \leq 0, \quad t \in \mathbb{R}.$$

Hence, $L_2 \in \mathfrak{T}((-\infty, +\infty))$, and we only need to use Theorem 2. The argument in the case when the second inequality (22) is satisfied is similar. \square

Let us note that inequality (22) is satisfied (in fact, as an equality) if, for instance,

$$r(t) \equiv -R^2, \quad p(t) = \frac{R \left(1 - c^2 e^{\frac{Rt}{2}} \right)}{1 + c^2 e^{\frac{Rt}{2}}}.$$

In this case equation $L_2 x = 0$ has solution

$$x(t) = e^{-\int_0^t \frac{R \left(1 - c^2 e^{\frac{Rs}{2}} \right)}{1 + c^2 e^{\frac{Rs}{2}}} ds} (> 0).$$

Note also that unlike the case of a differential equation with constant coefficients, in the case of variable coefficients condition $G_L \subset \mathfrak{N}$ is not necessary for disconjugacy of equation (20) on \mathbb{R} . For example, equation

$$(\overline{L})(t) := x'' + tx' + \left(\frac{t^2}{4} + \frac{1}{2} \right) x = 0,$$

having solution $x = e^{-\frac{t^2}{4}} > 0$ ($t \in \mathbb{R}$), is disconjugate on \mathbb{R} , although $G_{\overline{L}} \subset \mathfrak{D}$. The same is true for a more general equation

$$x'' + p(t)x' + \left(\frac{p^2(t)}{4} + \frac{1}{2} p'(t) \right) x = 0, \quad p'(t) > 0, \quad t \in \mathbb{R},$$

which has solution $x = e^{-\frac{1}{2} \int_0^t p(s) ds} > 0, \quad t \in \mathbb{R}$.

Using the last example and Theorem 2 we derive the following strengthening of Theorem 10.

Theorem 11. If function p is differentiable, $p'(t) \geq 0$ ($p'(t) \leq 0$) on \mathbb{R} and

$$q(t) \leq \frac{p^2(t)}{4} + \frac{1}{2} p'(t) \quad \left(q(t) \leq \frac{p^2(t)}{4} - \frac{1}{2} p'(t) \right) \quad (t \in \mathbb{R}),$$

then equation (20) is disconjugate on \mathbb{R} .

Remark 2. It is easy to show that a differential equation which is close to disconjugate differential equation, is disconjugate as well. The latter allows to weaken the assumptions on coefficient p , demanding only non-strict monotone increasing (decreasing).

10. Finally, let us consider equation

$$(24) \quad Lx := x'' + p(t)x' + q(t)x = 0 \quad (t \in (a, +\infty))$$

with coefficients continuous on $(a, +\infty)$. Substitution $t \rightarrow a + t^2$ reduces equation (24) to equation

$$(25) \quad Lx := x'' + p(a + t^2)x' + q(a + t^2)x = 0 \quad (t \in (-\infty, +\infty)).$$

Now, disconjugacy of equation (25) on \mathbb{R} is equivalent to disconjugacy of equation (24) on $(a, +\infty)$. By applying criteria for disconjugacy for equation (25), we get criteria for disconjugacy of equation (24) on $(a, +\infty)$.

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